CLOCK TIME AND ENTROPY

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1. What is fundamental in a quantum description of the world?

It is a matter of debate what the basic entities of the universe are in quantum theory. Expanding the metaphor used by Griffiths at this conference, one might imagine that the conversations of the three meals of the day could be restricted to three possible answers:

(a) Amplitudes (subject of lunch conversation). One might imagine formulating quantum theory in the morning and concluding at lunch that complex amplitudes were the fundamental entities. Each fine-grained history h_i might be assigned an amplitude

$$A[h_i] = e^{iS[h_i]}, \tag{1}$$

where $S[h_i]$ is the action (in units of $\hbar$) of that history (Feynman and Hibbs 1965). The fine-grained histories could then be combined into a weighted group of histories α (called a coarse-grained history) with amplitude

$$A[\alpha] = \sum_i w_\alpha [h_i] e^{iS[h_i]}, \tag{2}$$

where w_α is an α-dependent weight functional of the fine-grained histories h_i. For example, $A[\alpha]$ could be given by a path integral over a set of histories, so that w_α could be 1 if h_i is in the set and 0 if h_i is not. More generally, $w_\alpha[h_i]$ might be weighted by the complex amplitude for the initial configuration of the history h_i. If α consisted of all of the histories that reached a certain final configuration, and no others, $A[\alpha]$ would be what is called the wavefunction evaluated at that final configuration. A problem with these amplitudes as basic entities, however, is that it is not clear what direct interpretation they have.

(b) Probabilities for decohering sets of histories (subject of dinner conversation) (Griffiths 1984; Omnès 1988a, 1988b, 1988c, 1989; Gell-Mann and Hartle 1990; Hartle 1990a, 1990b, 1991). One might imagine interpreting the amplitudes in the afternoon so that the (unnormalized) probability associated with each coarse-grained history α is

$$p(\alpha) = |A(\alpha)|^2. \tag{3}$$

In order to get a set of probabilities obeying the usual axioms of probability, one then needs a set of α's such that for any pair $(\alpha \neq \alpha')$,

$$p(\alpha + \alpha') \equiv |A(\alpha) + A(\alpha')|^2 = p(\alpha) + p(\alpha') \equiv |A(\alpha)|^2 + |A(\alpha')|^2. \tag{4}$$

This means that the real part of the interference terms must vanish,

$$2\,\mathrm{Re}\,A(\alpha)A^*(\alpha') = A(\alpha)A^*(\alpha') + A^*(\alpha)A(\alpha') = 0. \tag{5}$$

More generally, one may have a decoherence functional $D(\alpha, \alpha')$ with

$$\mathrm{Re}\,D(\alpha, \alpha') = 0, \tag{6}$$

and then

$$p(\alpha) = D(\alpha, \alpha). \tag{7}$$

One then calls this set of α's a decohering set of (coarse-grained) histories. One says that probabilities are defined only for each member of such a decohering set. Then the viewpoint is that such probabilities are the fundamental entities of the quantum theory. Feeling that his task is basically completed, the theorist retires for the night.

(c) Testable conditional probabilities (subject of breakfast conversation) (Page and Wootters 1983; Page 1986a, 1986b, 1987, 1988, 1989, 1991a, 1991b, 1992a). After a good night's sleep, the theorist, able to think more clearly, realizes that even decohering sets of histories are too broad to be directly interpreted. Instead, it is a much narrower set of conditional probabilities that we, living within the universe, can test. So far as we know, each of these occurs on a single hypersurface (at a single "time"), and perhaps is also highly localized on that hypersurface.

That is, we cannot directly compare things at different times, but only different records at the same time. We cannot know the past except through its records in the present, so it is only present records that we can really test. For example, we cannot directly test the conditional probability that the electron has spin up at $t = t_f$, given that it had spin up at $t = t_i < t_f$, but only given that there are records at $t = t_f$ that we interpret as indicating the electron had spin up at $t = t_i$. Wheeler (1978) states this even more strongly: "the past has no existence except as it is recorded in the present."

This principle should perhaps also be extended to say that we cannot directly compare things at different locations either, so that all observations are really localized in space as well as in time. Certainly each of my observations seems to be localized within the spatial extent of my brain and the temporal extent of a single conscious moment, though it appears to be an exaggeration to say it is localized to a single spacetime point. Whether it is actually localized on a single hypersurface of a certain spatial extent is admittedly an open question, but since the duration of a single conscious moment is so much shorter than the apparent age of the universe, it seems reasonable and consistent with all we do know to idealize each observation as occurring at a single time, on a single spatial hypersurface.

If the quantum state of the universe on a spatial hypersurface is given by a density matrix ρ, the conditional probability of the result A , given a testable condition B, is

$$p(A|B) = \frac{\mathrm{Tr}\,(P_A P_B\,\rho\,P_B)}{\mathrm{Tr}\,(P_B\,\rho\,P_B)} = \frac{\mathrm{Tr}\,(P_B P_A P_B\,\rho)}{\mathrm{Tr}\,(P_B\,\rho)}, \tag{8}$$

where $P_A = P_A^\dagger = P_A^2$ and $P_B = P_B^\dagger = P_B^2$ are the corresponding projection operators. The testable conditional probabilities are then the subject of rational lunch conversation.

Of course, the theorist's job is not completed with the mere formulation of Eq. (8): one must discover the density matrix of the universe, formulate the projection operators, and calculate the quantities in Eq. (8). Then there is the question of which results and conditions are testable. One could avoid this problem by simply postulating that the conditional probabilities associated with all projection operators P_A and P_B are meaningful fundamental entities. However, most of these would not be readily interpretable, so there would seem to be little motivation not to go back to the complex amplitudes (1) and (2) as more fundamental.

A conclusion of these various meal discussions is that amplitudes seem to be the most basic entities, but that if one wishes a more restricted set that can be readily interpreted and tested, it does not seem to be going far enough to say they are the probabilities for decohering sets of histories. Testable conditional probabilities appear in reality to be restricted to a single hypersurface, not to histories, though even this restriction alone is almost certainly not enough.

2. Inaccessibility of coordinate time

As mentioned above, testable conditional probabilities (8) seem to be confined to a single "time" (e.g., a spatial hypersurface in canonical gravity). Yet they cannot depend on the value of the coordinate time labeling the hypersurface, which is completely unobservable. One can give three arguments for this fact:

(a) For a closed universe, the Wheeler-DeWitt equation, $H\psi = 0$, implies that ψ is independent of t.

(b) For an asymptotically-flat open universe, the long-range gravitational field provides a superselection rule for the total energy, just as the long-range Coulomb electric field provides a superselection rule for the total charge (Strocchi and Wightman 1974). This means that phases between states of different energy are unmeasurable, so the coordinate-time dependence of the density matrix ρ is not detectable (Page and Wootters 1983).

(c) In any case, one has no access to the coordinate time t, so one should average over this inaccessible variable to get a density matrix for the other variables (Page 1989). This time-averaged density matrix in the Schrödinger picture is

$$\overline{\rho} = \langle \rho(t) \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, \rho(t) \tag{9}$$

and in the Heisenberg picture is

$$\overline{\rho} = \langle e^{-iHt} \rho e^{iHt} \rangle \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, e^{-iHt} \rho\, e^{iHt}. \tag{10}$$

Without access to coordinate time t, we can only test conditional probabilities of the form

$$\begin{aligned} p(A|B) &= \frac{\langle \mathrm{Tr}\,[P_B P_A P_B\, \rho(t)] \rangle}{\langle \mathrm{Tr}\,(P_B\, \rho(t)) \rangle} = \frac{\mathrm{Tr}\,(P_B P_A P_B\, \overline{\rho})}{\mathrm{Tr}\,(P_B\, \overline{\rho})} = \frac{\langle \mathrm{Tr}\,(P_B P_A P_B e^{-iHt} \rho(0)\, e^{iHt}) \rangle}{\langle \mathrm{Tr}\,(P_B e^{-iHt} \rho(0)\, e^{iHt}) \rangle} \\ &= \frac{\mathrm{Tr}\,[\langle e^{iHt} P_B P_A P_B\, e^{-iHt} \rangle \rho(0)]}{\mathrm{Tr}\,[\langle e^{iHt} P_B\, e^{-iHt} \rangle \rho(0)]} = \frac{\mathrm{Tr}\,[\overline{P_B P_A P_B}\, \rho(0)]}{\mathrm{Tr}\,[\overline{P}_B\, \rho(0)]}, \end{aligned} \tag{11}$$

where in the Schrödinger picture

$$\overline{O} \equiv \langle e^{iHt} O e^{-iHt} \rangle \tag{12}$$

for an observable O without explicit time dependence, and in the Heisenberg picture

$$\overline{O} \equiv \langle O(t) \rangle = \langle e^{iHt} O(0) e^{-iHt} \rangle. \tag{13}$$

$$[H, \overline{O}] = 0, \tag{14}$$

so the observable $\overline{O}$ is stationary.

Thus testable conditional probabilities depend only on the stationary (time-averaged) density matrix $\overline{\rho}$, or alternatively, only on $\rho(0)$ and stationary observables like $\overline{P_B}$ and $\overline{P_B P_A P_B}$ $(\neq \overline{P_B}\,\overline{P_A}\,\overline{P_B})$. In a constrained Hamiltonian system like canonical quantum gravity, these observables, but not the individual projection operators, would commute with the constraints (e.g., the Wheeler-DeWitt operator).

3. Observable evolution as dependence on clock time

Although unobservable coordinate time cannot be part of the condition B of $p(A|B)$ given by Eq. (8) or (11), the reading of a physical clock can be. The dependence of $p(A|B)$ upon the clock reading is then the observable time evolution of the system.

For simplicity, consider the case where the condition is entirely the reading of a clock subsystem (C) with states $|\psi_C(T)\rangle$. Let

$$P_B \rightarrow P_T = |\psi_C(T)\rangle\langle\psi_C(T)| \otimes I_R, \tag{15}$$

where I_R is the unit operator for the rest (R) of the closed system (e.g., the rest of the universe). Then

$$p(A|T) = \frac{\mathrm{Tr}\,(P_T P_A P_T\, \overline{\rho})}{\mathrm{Tr}\,(P_T\, \overline{\rho})} \tag{16}$$

can vary with the clock time T.

In the special case that the clock does not interact with the rest of the system, so

$$H = H_C \otimes I_R + I_C \otimes H_R, \tag{17}$$

and in the case that the result A does not concern the clock, so

$$P_A = I_C \otimes P_{AR} \tag{18}$$

with P_{AR} acting only on R, then the choice of clock states of reading T as

$$|\psi_C(T)\rangle \equiv e^{-iH_C T}\, |\psi_C(0)\rangle \tag{19}$$

leads to the following familiar-looking equation for the conditional probability of A given the clock reading T :

$$p(A|T) = \operatorname*{tr}_R [P_{AR}\, \rho_R(T)]. \tag{20}$$

Here, the conditional density matrix for R at clock reading T is

$$\begin{aligned} \rho_R(T) &= \operatorname*{tr}_C (P_T\, \overline{\rho}\, P_T)/\mathrm{Tr}\,(P_T\, \overline{\rho}) \\ &= e^{-iH_R T} \rho_R(0) e^{iH_R T}, \end{aligned} \tag{21}$$

thus evolving (in the Schrödinger picture) according to the von Neumann equation with respect to the clock time T (Page and Wootters 1983).

In the more general case that the clock does interact with the rest of the system and/or if the clock states are not defined by Eq. (19) (which gives

$$P_T = e^{iHT} P_{T=0} e^{-iHT} \tag{22}$$

and has the somewhat undesirable consequence that $P_T\, P_{T'}$ is not necessarily zero for $T \neq T'$), then generically $p(A|T)$ can be obtained from Eq. (16), but it will not have the $\mathrm{tr}\,(P_{AR}\, \rho_R)$ form of Eq. (20) with a unitarily evolved density matrix $\rho_R(T)$. In this more general case clock time will not have been defined so that motion is so simple as Eqs. (20) and (21), in contrast to the criterion of Misner, Thorne, and Wheeler (1973).

Karel Kuchař (1992, and at lunch during this conference) has given two objections to the conditional probability interpretation outlined here:

(a) The application of the condition violates the constraints. Eq. (8) or (11) may be written as

$$p(A|B) = \mathrm{Tr}\,(P_A\, \rho_B), \tag{23}$$

where the conditional density matrix

$$\rho_B = P_B\, \rho\, P_B / \mathrm{Tr}\,(P_B\, \rho) \tag{24}$$

generally does not satisfy the constraints, e.g.,

$$[H, \rho_B] \neq 0, \tag{25}$$

if

$$[H, P_B] \neq 0. \tag{26}$$

However, ρ_B is merely a calculational device to simplify Eq. (8) or (11) to Eq. (23). One could instead rewrite Eq. (8) or (11) as

$$p(A|B) = \mathrm{Tr}\, \rho_{AB}, \tag{27}$$

where

$$\rho_{AB} = \langle P_A P_B\, \rho P_B P_A \rangle / \mathrm{Tr}\, \langle P_B\, \rho P_B \rangle \tag{28}$$

does obey the constraints.

(b) The conditional probability formula (8) does not give the right propagators. For example, if

$$P_{T_1} P_{T_2} = P_{T_2} P_{T_1} = 0, \tag{29}$$

$$P_{A_1} P_{T_1} = P_{T_1} P_{A_1} \neq 0, \tag{30}$$

$$P_{A_2} P_{T_2} = P_{T_2} P_{A_2} \neq 0, \tag{31}$$

then Kuchař wants $p(A_2, T_2 \,|\, A_1, T_1)$ to be the absolute square of the propagator (i.e., the transition probability) to go from A_1 at T_1 to A_2 at T_2, whereas the formula

$$p(A_2, T_2 \,|\, A_1, T_1) = \frac{\mathrm{Tr}\, (P_{A_2} P_{T_2} P_{A_1} P_{T_1}\, \rho\, P_{T_1} P_{A_1} P_{T_2} P_{A_2})}{\mathrm{Tr}\, (P_{A_1} P_{T_1}\, \rho P_{T_1} P_{A_1})} \tag{32}$$

gives zero.

In response, I would say that the absolute square of the propagator is

$$\big| \langle A_2 \text{ at } T_2 \mid A_1 \text{ at } T_1 \rangle \big|^2 = p(A_2 \text{ at } T_2 \mid A_1 \text{ at } T_1), \tag{33}$$

that is, the probability to have A_2 *at* T_2 if one had A_1 *at* T_1, whereas Eq. (32) gives

$$p(A_2, T_2 \mid A_1, T_1) = p(A_2 \text{ and } T_2 \mid A_1 \text{ and } T_1), \tag{34}$$

the probability that one has A_2 *and* T_2 if one has A_1 *and* T_1 on the same hypersurface. Since Eq. (29) says that T_1 and T_2 are mutually exclusive, this probability is zero: one cannot simultaneously have two distinct clock readings.

The conditional probability interpretation is dynamical in the sense that it gives the T dependence of $p(A|T)$ by Eq. (16). However, it is not so dynamical as Kuchař would like to give the transition probability of Eq. (33) directly. The reason it does not is that this quantity is not directly testable, since T_1 and T_2 are different conditions that cannot be imposed together. At $T = T_2$, one has no direct access to what happened at $T = T_1$. One must instead rely upon records, which can be checked at T_2.

One way that one could try to calculate the transition probability (33) theoretically is to try to construct a projection operator $P_{A_1T_1}$ $(\neq P_{A_1}P_{T_1})$ which, when applied to ρ, gives a density matrix

$$\rho_{A_1T_1} = P_{A_1T_1}\,\rho P_{A_1T_1}/\mathrm{Tr}\,(P_{A_1T_1}\,\rho) \tag{35}$$

which satisfies the constraints and which gives

$$p(A_1|T_1;\rho_{A_1T_1}) = \frac{\mathrm{Tr}\,(P_{T_1}P_{A_1}P_{T_1}\,\rho_{A_1T_1})}{\mathrm{Tr}\,(P_{T_1}\,\rho_{A_1T_1})} = 1. \tag{36}$$

Then one could say

$$p(A_2 \text{ at } T_2 \mid A_1 \text{ at } T_1) = p(A_2|T_2;\,\rho_{A_1T_1}). \tag{37}$$

However, it is not clear whether the answer is unique and whether it has certain desirable properties. More importantly, it is not directly testable, and therefore it is rather *ad hoc* which definition to propose for the transition probability.

4. The variation of entropy with clock time

The time asymmetry of the second law of thermodynamics should of course be expressed in terms of physical clock time rather than in terms of unobservable coordinate time (Page 1992b).

How to express entropy is more problematic. One mathematical expression is

$$S_T = -\mathrm{Tr}\,\rho_T\,\ln\rho_T, \tag{38}$$

$$\rho_T = P_T\,\overline{\rho}\,P_T/\mathrm{Tr}\,(P_T\,\overline{\rho}\,P_T). \tag{39}$$

If

$$\overline{\rho} = |\psi\rangle\langle\psi|, \tag{40}$$

a pure state, then

$$\rho_T = |\psi_T\rangle\langle\psi_T| \tag{41}$$

with

$$|\psi_T\rangle = P_T|\psi\rangle/\langle\psi|P_T|\psi\rangle^{1/2}, \tag{42}$$

another pure state, so $S_T = 0$.

But if $\overline{\rho}$ is not pure (e.g., is obtained from

$$\rho = |\psi\rangle\langle\psi| \tag{43}$$

pure but time dependent), then S_T can exceed zero and vary with the clock time T.

Example: Consider two coupled spin-$\frac{1}{2}$ systems in a vertical magnetic field, with Hamiltonian

$$\begin{aligned} H &= \frac{1}{4}\vec{\sigma}_C \cdot \vec{\sigma}_R + \frac{1}{2}\sigma_{C_z} \otimes I_R + \frac{1}{2} I_C \otimes \sigma_{R_z} + \frac{3}{4} I \\ &= 2\,|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + \frac{1}{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|), \end{aligned} \tag{44}$$

giving a triplet and a singlet degenerate with the lowest state of the triplet. Take the time-dependent pure state

$$|\psi\rangle = 0.1\sqrt{5}\big[(e^{-it}+1)|\uparrow\downarrow\rangle + (e^{-it}-1)|\downarrow\uparrow\rangle + 4\,|\downarrow\downarrow\rangle\big], \tag{45}$$

so the coordinate-time-averaged density matrix is

$$\begin{aligned} \overline{\rho} = 0.1\big(&|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + 2|\uparrow\downarrow\rangle\langle\downarrow\downarrow| + 2|\downarrow\downarrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow| \\ &- 2|\downarrow\uparrow\rangle\langle\downarrow\downarrow| - 2|\downarrow\downarrow\rangle\langle\downarrow\uparrow| + 8|\downarrow\downarrow\rangle\langle\downarrow\downarrow|\big). \end{aligned} \tag{46}$$

Let

$$P_T = \frac{1}{2}\big(|\uparrow\rangle + e^{-iT}\,|\downarrow\rangle\big)\big(\langle\uparrow| + e^{iT}\langle\downarrow|\big)_C \otimes I_R. \tag{48}$$

This leads to

$$\rho_T = \rho_{TC} \otimes \rho_{TR} \tag{49}$$

with

$$\rho_{TC} = \frac{1}{2}\big(|\uparrow\rangle + e^{-iT}|\downarrow\rangle\big)\big(\langle\uparrow| + e^{iT}\langle\downarrow|\big)_C, \tag{50}$$

$$\rho_{TR} = (10 + 4\cos T)^{-1}\big[|\uparrow\rangle\langle\uparrow| - 2|\uparrow\rangle\langle\downarrow| - 2|\downarrow\rangle\langle\uparrow| + (9 + 4\cos T)|\downarrow\rangle\langle\downarrow|\big]. \tag{51}$$

Therefore,

$$S(\rho(t)) = 0, \quad S(\overline{\rho}) = \ln 10 - 1.8 \ln 3 \approx 0.3251, \tag{52}$$

$$S_T = S(\rho_T) = \ln\left(\frac{10 + 4\cos T}{\sqrt{5 + 4\cos T}}\right) - \frac{2\sqrt{1 + (2 + \cos T)^2}}{5 + 2\cos T} \times \ln\left(\frac{5 + 2\cos T + 2\sqrt{1 + (2 + \cos T)^2}}{\sqrt{5 + 4\cos T}}\right), \tag{53}$$

which varies in a rather complicated way with clock time T.

Even when ρ_T is pure and gives $S_T = 0$, we may divide the system into subsystems and add the entropy of each subsystem density matrix (which ignores the information or negentropy of the correlations between the subsystems):

$$S_{T,\mathrm{coarse}} = -\sum_{i=1}^{n} \mathrm{tr}\, \rho_{T_i} \ \ln\ \rho_{T_i}, \tag{54}$$

$$\rho_{T_i} = \mathop{\mathrm{tr}}_{j \neq i} P_T\, \overline{\rho} P_T / \mathrm{Tr}\,(P_T\, \overline{\rho}). \tag{55}$$

Example: Consider three spin-$\frac{1}{2}$ systems in a vertical magnetic field, with Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\sigma_{C_z} \otimes I_{R1} \otimes I_{R2} + \frac{1}{2} I_C \otimes \sigma_{R1_z} \otimes I_{R2} \\ &\quad + \frac{1}{2} I_C \otimes I_{R1} \otimes \sigma_{R2_z} + \frac{1}{4} I_C \otimes \vec{\sigma}_{R1} \cdot \vec{\sigma}_{R2} + \frac{1}{4} I \\ &= 2|\uparrow\uparrow\uparrow\rangle\langle\uparrow\uparrow\uparrow| \\ &\quad + \frac{1}{2}\big(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle\big)\big(\langle\uparrow\uparrow\downarrow| + \langle\uparrow\downarrow\uparrow|\big) + |\downarrow\uparrow\uparrow\rangle\langle\downarrow\uparrow\uparrow| \\ &\quad - \frac{1}{2}\big(|\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle\big)\big(\langle\downarrow\uparrow\downarrow| - \langle\downarrow\downarrow\uparrow|\big) - |\downarrow\downarrow\downarrow\rangle\langle\downarrow\downarrow\downarrow|, \end{aligned} \tag{56}$$

so the first spin acts as a noninteracting clock and the remaining two have the same spin-spin coupling as the previous example. Take a pure-state linear combination of the zero-energy eigenvectors,

$$|\psi\rangle = \frac{1}{2}\big(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\big), \tag{57}$$

so Eq. (40) holds. Let P_T be given by Eq. (48) again, except that now I_R is the 4×4 unit matrix for the last two spins. This gives ρ_T of the pure form (41) with

$$|\psi_T\rangle = 2^{-1/2}\big(|\uparrow\rangle + e^{-iT}|\downarrow\rangle\big)_C \otimes \frac{1}{2}\big[(e^{iT}+1)|\uparrow\downarrow\rangle + (e^{iT}-1)|\downarrow\uparrow\rangle\big]_R, \tag{58}$$

so ρ_T again factorizes as in Eq. (49) between the clock and the rest, with ρ_{TC} given by Eq. (50) and

$$\begin{aligned}\rho_{TR} = \frac{1}{2}\big[&(1+\cos T)|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - i\sin T|\uparrow\downarrow\rangle\langle\downarrow\uparrow| \\ &+ i\sin T|\downarrow\uparrow\rangle\langle\uparrow\downarrow| + (1-\cos T)|\downarrow\uparrow\rangle\langle\downarrow\uparrow|\big],\end{aligned} \tag{59}$$

a pure state which does not factorize between the last two spins. The reduced density matrices of the two spins are

$$\rho_{TR1} = \frac{1}{2}I_{R1} + \frac{1}{2}\cos T\ \ (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_{R1}\,, \tag{60}$$

$$\rho_{TR2} = \frac{1}{2}I_{R2} - \frac{1}{2}\cos T\,(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)_{R2}\,, \tag{61}$$

each with eigenvalues

$$p_1 = \cos^2\frac{1}{2}T, \qquad p_2 = \sin^2\frac{1}{2}T. \tag{62}$$

Hence, the coarse-grained entropy given by Eq. (54) is

$$\begin{aligned}S_{T,\mathrm{coarse}} &= S(\rho_{TC}) + S(\rho_{TR1}) + S(\rho_{TR2}) \\ &= -2\cos^2\frac{1}{2}T\,\ln\cos^2\frac{1}{2}T - 2\sin^2\frac{1}{2}T\,\ln\sin^2\frac{1}{2}T,\end{aligned} \tag{63}$$

oscillating between 0 at $T = n\pi$ and $2\ln 2$ at $T = (n+1/2)\pi$.

Although they can vary with clock time, the mathematical and coarse-grained entropies defined by Eqs. (38) and (54) above are not really observables. One would like a truly observable entropy operator $\widehat{S}$ and then define the clock-time-dependent entropy as its conditional expectation value:

$$S_{T,\mathrm{observable}} = E(\widehat{S}|T) = \mathrm{Tr}\,(P_T\widehat{S}P_T\,\overline{\rho})/\mathrm{Tr}\,(P_T\,\overline{\rho}). \tag{64}$$

Can one find suitable definitions of $\widehat{S}$ and T so that $S_{T,\mathrm{observable}}$ increases fairly monotonically with T for the actual density matrix ρ of the universe?

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Discussion

Karel Kuchař: You always apply the conditional probability formula to calculate the conditional probability of a projector at a single instant of an internal clock time. You never apply it to answering the fundamental DYNAMICAL question of the internal Schrödinger interpretation, namely, "If one finds the particle at Q' at the time T', what is the probability of finding it at Q'' at the time T''? " By your formula, that conditional probability differs from zero only if $T' = T''$ and $Q' = Q''$. In brief, your interpretation prohibits the time to flow and the system to move!

For me, this virtually amounts to a reductio ad absurdum of the conditional probability proposal. One can trace this feature back to the fact that the conventional derivation of the conditional probability formula amounts to the violation of the super-Hamiltonian constraint.

Don Page: At this time I cannot give much more of an answer than I did in my lecture (which is slightly expanded for the Proceedings). In my viewpoint, only quantities at a single instant of time are directly accessible, and so one cannot directly test the two-time probability you discuss. One could instead at one time test the conditional probability that the particle is at Q'', given that the time is T'' and that at this time there is a *record* indicating that the particle was at Q' at the time T'. However, there is no direct way to test whether the record is accurate, though one can check whether different records show consistency. After all, that is the only way we have to increase our confidence in the existence of historical events.

Karel Kuchař: Don, you are the first person I met who simultaneously believes in the existence of many worlds and is a solipsist of an instant.

Don Page: I believe that different instants, i.e., different clock times, are actually examples of the different worlds. They all exist, but each observation, and its associated conditional probability, occurs at one single time (assuming that the condition includes or implies a precise value of the clock time in question). We can only directly observe and be aware of the world, and the time, in and at which we exist, though the correlations in memories and other structures we observe in one world give indirect evidence of other worlds, and other clock times, in the full quantum state of the universe. I do not deny the existence of historical events at different instants of clock time, but I do not believe that they have conditional probabilities (given our conditions here and now) exactly equal to unity, or even that they can be precisely assigned any probabilities at all, say in the sense of Eq. (7) when Eq. (6) holds. In any case, any imputed probabilities for events in the past or future cannot be directly tested.